3401,

1) a) $y'' - 3y' + 2y = f$; $y(0) = y'(0) = 0$. CF is $e^{\lambda x}$, $\lambda^2 - 3\lambda + 2 = (\lambda -$

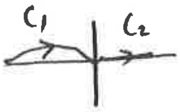
Try $y = Ae^{x} + Be^{2x}$, $y' = Ae^{x} + 2Be^{2x}$ where $A'e^{x} + B'e^{2x} = 0$ ①, $y'' = A'e^{x} +$ $+ Ae^{x} + 4Be^{2x}$. Substitution gives $A'e^{x} + 2B'e^{2x} = f(x)$ ②, ②-① $\Rightarrow$ $B' = f(x)e^{-2x}$ & so ① $\Rightarrow$ $A' = -f(x)e^{-x}$ $\Rightarrow$ $y = \alpha e^{x} + \beta e^{2x} + \int_0^x f(u)\{e^{2(x-u)} - e^{(x-u)}\}\,du$

$y(0) = 0 \Rightarrow \alpha + \beta = 0$, $y'(0) = 0 \Rightarrow \alpha + 2\beta + f(0)\{1-1\} = 0 \Rightarrow \alpha = \beta = 0$.

b) $xy'' + (6-x)y' - 5y = 0$. Substitution gives $\int_c e^{xt} f(t)\, x(t^2 - t)\,dt + \int_{c_i} e^{xt} f(t)(6t-5)\,dt = 0$

$\Rightarrow \left[e^{xt} f(t)\, t(t-1)\right]_{c_i} + \int_{c_i} e^{xt}\left[f(6t-5) - f(2t-1) - f'\,t(t-1)\right] dt = 0$. Set $[\;] = 0$

$\frac{f'}{f} = \frac{4t-4}{t(t-1)} = \frac{4}{t}$, $f = t^4$ & $y = \int_{c_i} t^4 e^{xt}\,dt$ where $\left[e^{xt} t^5 (t-1)\right]_{c_i} = 0$



$y_1(x) = \int_0^\infty t^4 e^{-xt}\,dt$, divergent as $x \to 0$, finite as $x \to \infty$ $\approx 4!/x^5$

$y_2(x) = \int_0^1 e^{xt} t^4\,dt$, divergent as $x \to \infty$, as $y_2 = e^{x}\int_0^1 e^{-xu}(1-u)^4\,du \sim \frac{e^{x}}{x}$

finite at $x = 0$

Only linear combination finite at $x = 0$ & ∞ is $0y_1 + 0y_2 = 0$.

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2) $\dot{x} = y - x = P$, $\dot{y} = f(x) - y = Q$

a) Bendixson's states periodic solutions not possible in regions where $P_x + Q_y$ ...
Here. $P_x + Q_y = -1 - 1 = -2 < 0$. So no periodic solutions

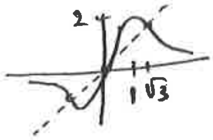
b) Critical points are where vertical nullcline, $dx/dt = 0$ ie $y = x$ & horizontal nullcline, $dy/dt = 0$ ie $y = f(x)$ meet ie $y = x = f(x)$

c) At a critical point the Jacobian is $\begin{pmatrix} P_x & P_y \\ Q_x & Q_y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ f' & -1 \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ s^2 & -1 \end{pmatrix}$. The eigenvalues satisfy $(-1-\lambda)^2 = s^2 \Rightarrow \lambda = -1 \pm s$. If $s < 1$ both are -ve & real, so we have a stable node. If $s > 1$, they differ in sign & we have a saddle point.

d) If $f'(a) < 0$, s is complex, $s = i\sigma$ & λ is complex with negative real part. The points are stable spiral points

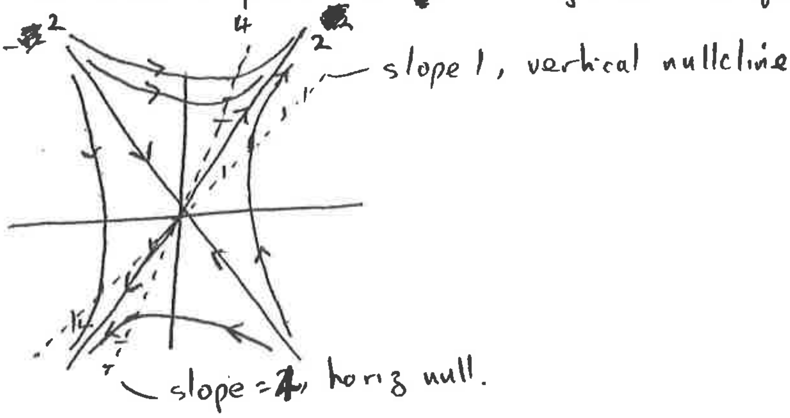
e) $f(x) = \dfrac{4x}{1+x^2}$, $f'(0) = 4$

$f'(x) = 4\dfrac{(1-x^2)}{(1+x^2)^2}$, $f'(a) < 0$ for $a = \sqrt{3}$



As f is odd, continuous & $f'(0) > 1$ we have three critical points. Two spiral points at (a,a) with $a = 4a/1+a^2 \Rightarrow a = \pm\sqrt{3}$ & a saddle at $x = 0$

Near $x = 0, y = 0$, $\dfrac{dx}{dt} = y - x$, $\dfrac{dy}{dt} = 4x - y$, $\lambda = -1 \pm \sqrt{4}$ & we have a saddle point. Solutions $y = mx$ require $dy/dx = m = \dfrac{4-m}{m-1} \Rightarrow m = \pm\sqrt{4} = 2$



. If $y > x$ $\dot{x} > 0$ & trajectories traversed with $x\uparrow$

3) $\ddot{x} + x + \varepsilon x^2 = 0$

a) $x = A\cos\theta + \varepsilon x_1 ;\quad \theta = (1 + \varepsilon n_1 \dots)t$

$(1 + 2\varepsilon n_1 \dots)(-A\cos\theta + \varepsilon x_1'') + A\cos\theta + \varepsilon A^2\cos^2\theta = 0$

$\Rightarrow x_1'' + x_1 = 2n_1 A\cos\theta - A^2\cos^2\theta$

For x_1 to be bounded & solution periodic the Fourier coefficient of $\cos\theta$ & $\sin\theta$ on r.h.s must be zero. So $\int_{-\pi}^{\pi} 2n_1 A\cos^2\theta - A^2\cos^3\theta \, d\theta = 0$ ie $2\pi A n_1 = A^2\int_{-\pi}^{\pi}\cos^3\theta d\theta$

Also $\int_{-\pi}^{\pi} 2n_1 A\cos\theta\sin\theta - A^2\cos^2\theta\sin\theta d\theta = 0$, which is true as integrand is odd.

However, $\int_{-\pi}^{\pi}\cos^3\theta d\theta = 0$, so $n_1 = 0$ & period $= 2\pi/1+\varepsilon n_1 + \dots = 2\pi + o(\varepsilon)$

& $x_1'' + x_1 = -A^2\cos^2\theta = -A^2\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right) \Rightarrow x_1 = A^2\left(\frac{\cos 2\theta}{6} - \frac{1}{2}\right)$ ignoring the C.F. wlog (similar seen)

b) If $x = A\cos\theta + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ & $\theta = (1 + \varepsilon^2 n_2 + \dots)t$, then

$x_2'' + x_2 = +2n_2 A\cos\theta - 2A^3\cos\theta\left(\frac{\cos 2\theta}{6} - \frac{1}{2}\right) - bA^3\cos^3\theta$.

We need n_2 to be zero for period to be $2\pi + o(\varepsilon^2)$ & the Fourier coefficient of $\cos\theta$ on r.h.s to be zero also ie

$$\int_{-\pi}^{\pi} 2\underbrace{n_2}_{0} A\cos^2\theta - 2A^3\cos^2\theta\left(\frac{\cos 2\theta}{6} - \frac{1}{2}\right) - bA^3\cos^4\theta \, d\theta = 0$$

As $\cos 2\theta = 2\cos^2\theta - 1$ & with $I_n = \int_{-\pi}^{\pi}\cos^n\theta d\theta$,

$\frac{2}{6}\cdot 2I_4 - \frac{2}{6}I_2 - I_2 + bI_4 = 0$, $b = \frac{4}{3} I_2/I_4 - \frac{2}{3}$

$I_2 = \pi$, $I_4 = \int_{-\pi}^{\pi}\cos^2\theta(1 - \sin^2\theta)d\theta = I_2 - \int_{-\pi}^{\pi}\frac{1}{4}\sin^2 2\theta \, d\theta = I_2 - \frac{1}{4}\cdot\pi$
$= 3\pi/4$

So $I_2/I_4 = \pi/3\pi/4 = 4/3$ & $b = 16/9 - 2/3 = 10/9$

4) a) $\ddot{x} + x = \epsilon f(x, \dot{x})$,

$T = \epsilon t$, $x = x_0(t,T) + \epsilon x_1(t,T)$ | $x_{0tt} + x_0 = 0 \Rightarrow x_0 = A(T) \sin \ldots (t)$

& $x_{1tt} + x_1 = f(x_0, \dot{x}_0) - 2\frac{\partial}{\partial T} x_{0t}$

$= f(A\sin\chi, A\cos\chi) - 2\frac{\partial}{\partial T} A\cos\chi$

To keep solution periodic, ie x_1 bounded, the Fourier coefficient of $\sin\chi$ & $\cos\chi$ on r.h.s. should be zero. As $\frac{\partial}{\partial T} A\cos\chi = A_T\cos\chi - \Phi_T A\sin\chi$, the $\cos\chi$ coefficient gives

$$0 = \int_{-\pi}^{\pi} \cos\chi\, f(A\sin\chi, A\cos\chi) - 2A_T \int_{-\pi}^{\pi} \cos^2\chi\, d\chi \Rightarrow A_T = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos\chi\, f(A\sin\chi, A\cos\chi)\, d\chi$$

& $\sin\chi$

$$0 = \int_{-\pi}^{\pi} \sin\chi\, f(A\sin\chi, A\cos\chi) + 2\Phi_T A \int_{-\pi}^{\pi} \sin^2\chi\, d\chi \Rightarrow A\,\partial\Phi/\partial T = -\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin\chi\, f(\,,\,)\, d\chi$$

(seen)

b) For $f(x,\dot{x}) = \dot{x}(a^2 - x^2)$,

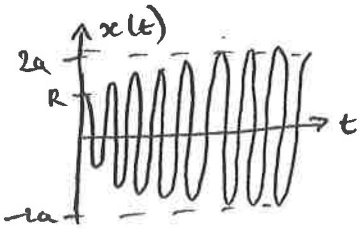
$$A\frac{d\Phi}{dT} = -\frac{1}{2\pi}\int_{-\pi}^{\pi} A\cos\chi\,(a^2 - A^2\sin^2\chi)\,.\sin\chi\, d\chi = 0 \Rightarrow \Phi_T = 0,\quad \Phi = \pi/2 \text{ fixes the boundary conditions}$$

$$A_T = \frac{1}{2\pi}\int_{-\pi}^{\pi} A\cos^2\chi\,(a^2 - A^2\sin^2\chi)\, d\chi = \frac{1}{2\pi}\left\{ Aa^2\pi - \frac{1}{4}A^3\pi \right\}$$

If $Q = A^2$, $\quad Q_T = a^2 Q - \frac{1}{4}Q^2 = \frac{4a^2Q - Q^2}{4}$, $\quad \frac{4\, dQ}{4a^2Q - Q^2} = dT = \frac{1}{a^2}\left(\frac{1}{Q} + \frac{1}{4a^2 - Q}\right)$

$$\Rightarrow a^2 T = \ln\left(\frac{Q}{4a^2 - Q}\cdot\frac{4a^2 - R^2}{R^2}\right) \Rightarrow \frac{A^2}{4a^2 - A^2} = \frac{R^2}{4a^2 - R^2}e^{a^2T},\quad \frac{4a^2}{A^2} - 1 = \left(\frac{4a^2}{R^2} - 1\right)e^{-a^2T}$$

$$\Rightarrow \frac{4a^2R^2}{A^2} = R^2 + (4a^2 - R^2)e^{-a^2T} \quad \& \text{ so } \quad x(t) = A\cos t = \frac{2aR\cos t}{\sqrt{R^2 + (4a^2 - R^2)e^{-a^2T}}}$$



(similar seen)

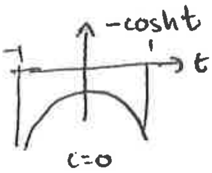
5) a)

i) $\int_0^\infty \frac{e^{-xt}}{1+t^2}\,dt \sim \frac{1}{x}\sum_0^\infty \frac{(-1)^n(2n)!}{x^{2n}}$ as $\frac{1}{1+t^2} \sim \sum_0^\infty (-1)^n t^{2n}$ as $t \to 0$

using Watsons

$\int_0^T e^{-xt} f(t)\,dt \sim \frac{1}{x}\left(\sum_0^\infty \frac{f_n\, n!}{x^n}\right)$ if $f \sim \sum_0^\infty f_n t^n$ as $t \to 0$

ii) $\int_{-1}^{1} \frac{e^{-x\cosh t}}{\cosh t}\,dt \sim e^{-x}\int_{-\infty}^{\infty} \frac{e^{-xt^2/2}}{\cosh 0}\,dt = e^{-x}\sqrt{\frac{\pi}{x/2}}$



or use formula for Laplace integrals $\int_a^b e^{x\varphi(t)} f(t)\,dt \sim f(c)\,e^{x\varphi(c)}\sqrt{\frac{\pi 2}{x|\varphi''(c)|}}$

$\varphi'(c)=0$

$\varphi''(c)<0$

iii) Use the method of stationary phase

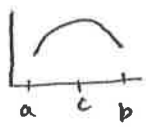
phase xt^2 stationary at end point.

$$\int_0^{\pi/4} \frac{\cos(xt^2)}{1+\tan t}\,dt = \frac{1}{2}\int_0^{\pi/4} \frac{e^{ixt^2}+e^{-ixt^2}}{1+\tan t}\,dt \sim \frac{1}{2}\left\{\frac{1}{2}\left(e^{ix0}e^{i\pi/4}\sqrt{\frac{2\pi}{x2}} + e^{ix0}e^{-i\pi/4}\sqrt{\frac{2\pi}{x|2|}}\right)\right\}$$

$$\left\{\int_a^b e^{ix\varphi(t)} f(t)\,dt \sim e^{ix\varphi(c)}\, e^{i\,\mathrm{sgn}(\varphi''(c))\pi/4}\sqrt{\frac{2\pi}{x|\varphi''(c)|}}\, f(c),\quad \varphi(t)=\pm t^2,\ \varphi''(0)=\pm 2,\ f(t)=1/(1+\tan t),\ f(0)=1\right\}$$

$$\sim \frac{1}{4}\sqrt{\frac{2\pi}{2x}}\cdot 2\cos\pi/4 = \frac{1}{2\sqrt{2}}\sqrt{\frac{\pi}{x}} = \sqrt{\frac{\pi}{8x}}$$

b) $$I(x)=\int_a^b e^{x\varphi(t)} f(t)\,dt \sim \int_a^b e^{x\varphi(c) + x\frac{\varphi''(c)}{2}(t-c)^2 \ldots}\cdot\left\{f(c) + (t-c)f'(c) + \frac{1}{2}f''(c)(t-c)^2 \ldots\right\}dt$$



put $(t-c) = \sqrt{\frac{2}{|\varphi''(c)|x}}\,u$ & as $\varphi''(c)<0$

$$I(x) \sim e^{x\varphi(c)}\int_{-\infty}^{\infty} e^{-u^2}\left\{0 + \overbrace{f'(c)\,u\sqrt{\frac{2}{x|\varphi''(c)|}}}^{\text{odd}} + \frac{1}{2}f''(c)\,u^2\cdot\frac{2}{|\varphi''(c)|x}\ldots\right\}\sqrt{\frac{2}{|\varphi''(c)|x}}\,du$$

$$\sim e^{x\varphi(c)}\cdot\frac{1}{2}f''(c)\left(\frac{2}{|\varphi''(c)|x}\right)^{3/2}\int_{-\infty}^{\infty} u^2 e^{-u^2}\,du$$

$$\int_{-\infty}^{\infty} u^2 e^{-u^2}\,du = 2\int_0^{\infty} u^2 e^{-u^2}\,du = \cancel{2}\int_0^{\infty} v\,e^{-v}\frac{dv}{\cancel{2}\sqrt{v}} = \left(\tfrac{1}{2}\right)! = \tfrac{1}{2}\left(-\tfrac{1}{2}\right)! = \tfrac{1}{2}\sqrt{\pi}$$

$v=u^2$, $dv=2u\,du$

$$I(x) \sim e^{x\varphi(c)}\cdot f''(c)\sqrt{\frac{\pi}{2(x|\varphi''(c)|)^3}}$$